

Large N limit of integrable models

M. Olshanetsky

Institute of Theoretical and Experimental Physics, Moscow, Russia,

e-mail olshanet@gate.itep.ru

Dedicated to the sixty-fifth birthday of Sergey Novikov

Abstract

We consider a large N limit of the Hitchin type integrable systems. The first system is the elliptic rotator on GL_N that corresponds to the Higgs bundle of degree one over an elliptic curve with a marked point. This system is gauge equivalent to the N -body elliptic Calogero-Moser system, that is derived from the Higgs bundle of degree zero over the same curve. The large N limit of the former system is the integrable rotator on the group of the non-commutative torus. Its classical limit leads to the integrable modification of 2d hydrodynamics on the two-dimensional torus. We also consider the elliptic Calogero-Moser system on the group of the non-commutative torus and consider the systems that arise after the reduction to the loop group.

Contents

1	Introduction	2
2	The Lie algebra of the non-commutative torus	3
3	2d-hydrodynamics on \mathcal{A}_θ	5
4	$SL(N, \mathbb{C})$-elliptic rotator	7
5	Elliptic rotator on \mathcal{A}_θ	9
6	Elliptic Calogero-Moser system on \mathcal{A}_θ	13
7	Conclusion	18
8	Acknowledgments	19

9	Appendix	19
9.1	Appendix A. Hitchin systems on an elliptic curve.	19
9.2	Appendix B. Elliptic functions.	21

1 Introduction

In this paper we analyze two related integrable models - a modified integrable two-dimensional hydrodynamics on a torus and the large N limit of the elliptic Calogero-Moser system (ECMS) with spin. The former system is also the large N limit of the integrable $GL(N, \mathbb{C})$ elliptic rotator (ER) proposed in Ref. [1]. Integrable models are among the main scientific interests of Sergey Novikov and his contribution to this subject is widely recognized.

It was established in Ref. [2] that for finite N ECMS is gauged equivalent to ER. The both systems are particular examples of the Hitchin construction [3, 5, 4]. Namely, they are derived from the Higgs bundles of rank N over an elliptic curve. The corresponding bundle for ECMS has degree zero while the bundle corresponding to ER has degree one. The gauge equivalence is determined by the upper modification that transforms the trivial bundle into the bundle of degree one.

We analyze the both systems in a similar way trying to establish the structures that were already known for the open finite-dimensional Toda chain. Namely, for the open finite-dimensional Toda chain there exists a limit to the infinite chain [6], reduction of the infinite chain to the periodic, and the dispersionless version of the infinite chain [7, 8].

Here we consider a special limit $N \rightarrow \infty$ that corresponds to the passage from $GL(N, \mathbb{C})$ to the infinite-dimensional group of the non-commutative torus (NCT). We consider first ER¹. This system is an example of the integrable Euler-Arnold tops (EAT) on the group $SL(N, \mathbb{C})$. EAT are Hamiltonian systems defined on coadjoint orbits of groups [10]. Particular examples of such systems are the Euler top on $SO(3)$, its integrable $SO(N)$ generalization [11, 12] and the hydrodynamics of the ideal incompressible fluid on a space M . The corresponding group of the latter system is $SDiff(M)$. We consider here the case when M is a torus T^2 . EAT are completely determined by their Hamiltonians, since the Poisson structure is fixed to be related to the Kirillov-Kostant form on the coadjoint orbits. The Hamiltonians are determined by the inertia-tensor operator \mathfrak{J} mapping the Lie algebra \mathfrak{g} to the Lie coalgebra \mathfrak{g}^* . Special choices of \mathfrak{J} lead to completely integrable systems (see review [13]). In the case of the 2d hydrodynamics \mathfrak{J} has the form of the Laplace operator and it turns out that the theory is non-integrable [14]. One of the goals of this paper are integrable EAT related to $SDiff$. Some integrable models related to $SDiff$ were considered in [8, 15, 16].

Integrable models on $SDiff(M)$ can be described as the classical limit (dispersionless limit) of integrable models when the commutators in the Lax equations are replaced by Poisson brackets. This approach was proposed in Ref. [17, 18], and later developed in numerous publications (see, for example, the review [19]).

Here we use the same strategy defining an integrable system on the non-commutative torus (NCT) and then taking the classical limit to $SDiff(T^2)$. In analogy with ER on $GL(N, \mathbb{C})$ we consider a special limit $N \rightarrow \infty$ of $GL(N, \mathbb{C})$ that leads to the group G_θ of the NCT, where θ is the non-commutative parameter.² The Hamiltonian is defined by the inertia-tensor operators depending on the module τ , $Im\tau > 0$ of an elliptic curve. This curve is the basic spectral curve in the Hitchin description of the model. The group G_θ is defined as the set of invertible elements

¹This part is an extended version of the talk [9].

²For Manakov's top $\lim N \rightarrow \infty$ was considered in Ref.[20].

of the NCT algebra \mathcal{A}_θ . It can be embedded in $\text{GL}(\infty)$ and in this way G_θ can be described as a special limit of $\text{GL}(N, \mathbb{C})$. We define a family of EAT on G_θ parameterized by τ . Then, we construct the Lax operator with the spectral parameter on an elliptic curve with the same parameter τ .

In the classical limit $\theta \rightarrow 0$, $G_\theta \rightarrow \text{SDiff}(T^2)$ and the inertia-tensor operator \mathfrak{J} takes the form $\bar{\partial}^2$. The conservation laws survive in this limit while commutators in the Lax hierarchy become the Poisson brackets. It turns out that the classical limit is essentially the same as the rational limit of the basic elliptic curve, so that the product of the Planck constant θ and the half periods of the basic curve are constant.

We also construct ECMS system related to NCT. In both cases of we discuss the systems that arise after the reduction to the loop algebra $\hat{L}(\text{GL}(N, \mathbb{C}))$.

2 The Lie algebra of the non-commutative torus

Here we reproduce some basic results about NCT and related to it the Lie algebra \sin_θ [21].

1. *Non-commutative torus.*

NCT \mathcal{A}_θ is an unital algebra with two generators (U_1, U_2) that satisfy the relation

$$U_1 U_2 = \mathbf{e}(-\theta) U_2 U_1, \quad \mathbf{e}(\theta) = e^{2\pi i \theta}, \quad \theta \in [0, 1). \quad (2.1)$$

Elements of \mathcal{A}_θ are the double sums

$$\mathcal{A}_\theta = \{x = \sum_{m,n \in \mathbb{Z}} a_{m,n} U_1^m U_2^n, \quad a_{m,n} \in \mathbb{C}\},$$

where $a_{m,n}$ are elements of the ring \mathfrak{S} of the rapidly decreasing sequences on \mathbb{Z}^2

$$\mathfrak{S} = \{a_{m,n} \mid \sup_{m,n \in \mathbb{Z}} (1 + m^2 + n^2)^k |a_{m,n}|^2 < \infty, \text{ for all } k \in \mathbb{N}\}. \quad (2.2)$$

The trace functional $\text{tr}(x)$ on \mathcal{A}_θ is defined as

$$\text{tr}(x) = a_{00}. \quad (2.3)$$

The dual space to \mathfrak{S} is

$$\mathfrak{S}' = \{s_{k,j} \mid \sum_{m+j=0, \quad n+k=0} a_{m,n} s_{kj} < \infty, \quad a_{m,n} \in \mathfrak{S}\}. \quad (2.4)$$

The associative algebra \mathcal{A}_θ can be regarded as the quantization of the commutative algebra of smooth functions on the two-dimensional torus

$$T^2 = \{\mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}\} \sim \{0 < x \leq 1, 0 < y \leq 1\}. \quad (2.5)$$

by means of the identification

$$U_1 \rightarrow \mathbf{e}(x), \quad U_2 \rightarrow \mathbf{e}(y), \quad (2.6)$$

where the multiplication of functions on T^2 is the Moyal multiplication:

$$(f * g)(x, y) := fg + \sum_{n=1}^{\infty} \frac{(\pi\theta)^n}{n!} \varepsilon_{r_1, s_1} \cdots \varepsilon_{r_n, s_n} (\partial_{r_1 \dots r_n}^n f)(\partial_{s_1 \dots s_n}^n g). \quad (2.7)$$

The trace functional (2.3) in the Moyal identification is the integral

$$\text{tr} f = \int_{\mathcal{A}_\theta} f dx dy = f_{00}. \quad (2.8)$$

Another representation of \mathcal{A}_θ is defined by the operators, that act on the space of Schwartz functions on \mathbb{R}

$$U_1 \rightarrow \mathbf{e}(-2\pi\theta\partial_\varphi), \quad U_2 \rightarrow \exp(i\varphi). \quad (2.9)$$

Finally, we can identify U_1, U_2 with matrices from $\text{GL}(\infty)$. We define $\text{GL}(\infty)$ as the associative algebra of infinite matrices $a_{jk}E_{jk}$ such that

$$\sup_{j,k \in \mathbb{Z}} |a_{jk}|^2 |j-k|^n < \infty \quad \text{for all } n \in \mathbb{N}.$$

Consider the following two matrices from $\text{GL}(\infty)$

$$Q = \text{diag}(\mathbf{e}(j\theta)), \quad \Lambda = E_{j,j+1}, \quad j \in \mathbb{Z}.$$

We have the following identification

$$U_1 \rightarrow Q, \quad U_2 \rightarrow \Lambda. \quad (2.10)$$

2. sin-algebra

Define the following quadratic combinations of the generators

$$T_{m,n} = \frac{i}{2\pi\theta} \mathbf{e}\left(\frac{mn}{2}\theta\right) U_1^m U_2^n. \quad (2.11)$$

Their commutator has the form of the sin-algebra [21]

$$[T_{m,n}, T_{m',n'}] = \frac{1}{\pi\theta} \sin \pi\theta(mn' - m'n) T_{m+m', n+n'}. \quad (2.12)$$

We denote by \sin_θ the Lie algebra with the generators $T_{m,n}$ over the ring \mathfrak{S} (2.2)

$$\psi = \sum_{m,n} \psi_{m,n} T_{m,n}, \quad \psi_{m,n} \in \mathfrak{S}, \quad (2.13)$$

and by G_θ the group of invertible elements from \mathcal{A}_θ . The coalgebra \sin_θ^* is the linear space

$$\sin_\theta^* = \left\{ \mathcal{S} = \sum_{jk} s_{jk} T_{jk}, \quad s_{jk} \in \mathfrak{S}' \right\}.$$

In the Moyal representation (2.7) the commutator of \sin_θ has the form

$$[f(x, y), g(x, y)] = \{f, g\}^* := \frac{1}{\theta} (f * g - g * f). \quad (2.14)$$

The algebra \sin_θ has a central extension $\widehat{\sin}_\theta$. The corresponding additional term in (2.12) has the form of the star-brackets

$$(am + bn)\delta_{m,-m'}\delta_{n,-n'}, \quad a, b \in \mathbb{C}. \quad (2.15)$$

In other words, the commutator in $\widehat{\sin}_\theta$ takes the form

$$[f, g] = \{f, g\}^* + \mathbf{c} \frac{1}{4\pi^2} \int_{\mathcal{A}_\theta} f(a\partial_x g + b\partial_y g) .$$

3. *Loop algebra* $\hat{L}(\mathfrak{sl}(N, \mathbb{C}))$.

Let θ be a rational number $\theta = p/N$, where $p, N \in \mathbb{N}$ are mutually prime. In this case \mathcal{A}_θ has the ideal

$$I_N = \left\{ \sum c_{m,n}^{(l)} (U_1^m U_2^n - U_1^{m+Nl} U_2^n) = 0, \quad l = \overline{1, N} \right\} .$$

The factor-algebra \mathcal{A}_θ/I_N can be represented by embedding in $\mathrm{GL}(\infty)$. Represent an arbitrary element of GL_∞ as

$$\psi_{m,n} \mathbf{e} \left(\frac{mn}{N} \right) U_1^m U_2^n .$$

In the factor-algebra one has $\psi_{Ns+k,n} = \psi_{k,n}$. Then any element from \mathcal{A}_θ/I_N takes the form

$$\sum_{l \in \mathbb{Z}} a_{j,r}^{(l)} E_{j,j+Nl+r}, \quad j = \overline{1, N}, r = \overline{-N+1, N-1},$$

where $a_{j,r}^{(l)} = \sum_{k=1}^N \psi_{k,Nl+r} \mathbf{e} \left(\frac{kj}{N} \right)$. We put in correspondence the current from $L(\mathfrak{sl}(N, \mathbb{C}))$

$$g(t) = \sum_{l \in \mathbb{Z}} g_{j,r}^{(l)} E_{j,j+r} t^{Nl+r} .$$

After the gauge transform by $\mathrm{diag}(1, t, \dots, t^{N-1})$ we kill the factor t^r and then by replacing $w = t^N$ we come to the loop algebra with the principle gradation

$$g(w) = \sum_{l \in \mathbb{Z}} g^{(l)} w^l .$$

The central extension $\oint \mathrm{Tr}(g_1(w) \partial_w g_2(w) \frac{dw}{w})$ is proportional to the cocycle (2.15) for $a = 0$, $b = 1$. Here Tr is the trace in the fundamental representation of $\mathfrak{sl}(N, \mathbb{C})$.

3 2d-hydrodynamics on \mathcal{A}_θ

1. 2-d hydrodynamics

Let $\mathbf{v} = (V_x, V_y)$ be the velocity of the ideal incompressible fluid on a compact manifold M ($\dim(M) = 2$, $\mathrm{div} \mathbf{v} = 0$) and $\mathrm{curl} \mathbf{v} = \partial_x V_y - \partial_y V_x$ be its vorticity.³ The Euler equation for 2d hydrodynamics takes the form [10]

$$\partial_t \mathrm{curl} \mathbf{v} = \mathrm{curl}[\mathbf{v}, \mathrm{curl} \mathbf{v}] . \quad (3.1)$$

Define the stream function $\psi(x, y)$ as the Hamiltonian function generating the vector field \mathbf{v}

$$i_{\mathbf{v}} dx \wedge dy = d\psi .$$

In other words

$$V_x = \partial_y \psi, \quad V_y = -\partial_x \psi . \quad (3.2)$$

³For simplicity we assume that the measure on M is $dx \wedge dy$, though all expressions can be written in a covariant way.

Let \mathfrak{g} be the Poisson algebra of the stream functions $\mathfrak{g} = \{\psi\}$ on M defined up to constants $\mathfrak{g} \sim C^\infty(M)/\mathbb{C}$

$$\{\psi_{\mathbf{v}_1}, \psi_{\mathbf{v}_2}\} = -i_{\mathbf{v}_1} d\psi_2 = .$$

Consider the Lie algebra $SVectM$ of vector fields $\text{div} \mathbf{v} = 0$ on M . We have the following interrelation between the Lie algebras \mathfrak{g} and $SVect(M)$

$$\begin{aligned} \psi_{[\mathbf{v}_1, \mathbf{v}_2]} &= \{\psi_{\mathbf{v}_1}, \psi_{\mathbf{v}_2}\}, \\ 0 \rightarrow \mathbb{C}^2 &\rightarrow SVect(M) \rightarrow \mathfrak{g} \rightarrow 0, \end{aligned}$$

where the map $SVect(M) \rightarrow \mathfrak{g}$ is defined by (3.2) and the image of \mathbb{C}^2 is generated by the two fluxes $(c_1 \partial_1, c_2 \partial_2)$.

Let \mathfrak{g}^* be the dual space of distributions on M . The vorticity $\mathcal{S} = \text{curl} \mathbf{v}$ of the vector field \mathbf{v}

$$\mathcal{S} = -\Delta \psi$$

can be considered as an element from \mathfrak{g}^* . The Euler equation (3.1) in terms of the Poisson brackets has the form

$$\partial_t \mathcal{S} = \{\mathcal{S}, \psi\}, \text{ or } \partial_t \mathcal{S} = \{\mathcal{S}, \Delta^{-1} \mathcal{S}\}. \quad (3.3)$$

We can view (3.3) as the Euler-Arnold equation for the rigid top related to the Lie algebra \mathfrak{g} , where the Laplace operator is the map

$$\Delta : \mathfrak{g} \rightarrow \mathfrak{g}^*$$

that plays the role of the inertia-tensor. The phase space of the system is a coadjoint orbit of the group of the canonical transformations $\text{SDiff}(M)$. The equation (3.3) takes the form

$$\partial_t \mathcal{S} = \text{ad}_{\nabla H}^* \mathcal{S}, \quad (3.4)$$

where $\nabla H = \frac{\delta H}{\delta \mathcal{S}} = \psi$ is the variation of the Hamiltonian

$$H = -\frac{1}{2} \int_M \mathcal{S} \Delta^{-1} \mathcal{S} = \int_M \psi \Delta \psi. \quad (3.5)$$

There is infinite set of Casimirs defining the coadjoint orbits:

$$C_k = \int_M \mathcal{S}^k. \quad (3.6)$$

Consider a particular case, when M is a two-dimensional torus (2.5) equipped with the measure $-\frac{dx dy}{4\pi^2}$. In terms of the Fourier modes $s_{m,n}$ of the vorticity

$$\mathcal{S} = \sum_{m,n} s_{m,n} \mathbf{e}(-mx - ny)$$

the Hamiltonian (3.5) is

$$H = -\frac{1}{2} \sum_{m,n} \frac{1}{m^2 + n^2} s_{m,n} s_{-m,-n}, \quad (3.7)$$

and we come to the equation

$$\partial_t s_{m,n} = \sum_{j,k} \frac{jn - km}{j^2 + k^2} s_{jk} s_{m-j, n-k}. \quad (3.8)$$

2. 2d hydrodynamics on non-commutative torus.

We can consider the similar construction by replacing the Poisson brackets by the Moyal brackets (2.14) [22, 23]. Introduce the vorticity \mathcal{S} as an element of \sin_θ^*

$$\mathcal{S} = \sum_{m,n} s_{m,n} T_{-m,-n}. \quad (3.9)$$

The equation (3.3) takes the form

$$\partial_t \mathcal{S}(x, y) = \{\mathcal{S}(x, y), \Delta^{-1} \mathcal{S}(x, y)\}^*,$$

or for the Fourier modes

$$\partial_t s_{m,n} = \frac{1}{8\pi^3 \theta} \sum_{j,k} \frac{\sin(\pi \theta(jn - km))}{j^2 + k^2} s_{j,k} s_{m-j, n-k}. \quad (3.10)$$

This system is EAT on the group G_θ of invertible elements of \mathcal{A}_θ and the coadjoint orbits are defined by the same Casimirs (3.6) as for $\text{SDiff}(T^2)$. In the limit $\theta \rightarrow 0$ (3.10) reproduces (3.8).

4 $\text{SL}(N, \mathbb{C})$ -elliptic rotator

1. Elliptic rotator (ER) on $\text{SL}(N, \mathbb{C})$

Now we consider differential equations related to $\text{SL}(N, \mathbb{C})$ apriori not coming from the hydrodynamics. The elliptic $\text{SL}(N, \mathbb{C})$ -rotator is an example of EAT [10]. It is defined on $\mathfrak{sl}(N, \mathbb{C})^*$ and its phase space is a coadjoint orbit of $\text{SL}(N, \mathbb{C})$:

$$\mathcal{R}^{rot} = \{\mathcal{S} \in \mathfrak{sl}(N, \mathbb{C}), \quad \mathcal{S} = g^{-1} \mathcal{S}^{(0)} g\}. \quad (4.1)$$

The phase space \mathcal{R}^{rot} is equipped with the Kirillov-Kostant symplectic form

$$\omega^{rot} = \text{tr}(\mathcal{S}^{(0)} Dg g^{-1} \wedge Dg g^{-1}). \quad (4.2)$$

The Hamiltonian has the form

$$H^{rot} = -\frac{1}{2} \text{Tr}(\mathcal{S} J(\mathcal{S})), \quad (4.3)$$

where J is a linear operator on $\mathfrak{sl}(N, \mathbb{C})$. The inverse operator is called the inertia tensor. The equation of motion takes the form

$$\partial_t \mathcal{S} = [\mathcal{S}, J(\mathcal{S})]. \quad (4.4)$$

A special form of J provides the integrability of the system [1, 2]. Represent \mathcal{S} in the form

$$\mathcal{S} = -\frac{4\pi^2}{N^3} \sum_{m,n} S_{m,n} T_{-m,-n},$$

where $T_{m,n}$ is the basis of $\mathfrak{sl}(N, \mathbb{C})$, similar to the basis of the sin-algebra (2.11)

$$T_{m,n} = \frac{iN}{2\pi} \mathbf{e}\left(\frac{mn}{N}\right) Q_N^m \Lambda_N^n,$$

$$(m, n) \in \tilde{\mathbb{Z}}_N^{(2)} = \{(\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}) \setminus (0, 0)\}.$$

Here Λ_N, Q_N are defined by (A.7). Let

$$J(\mathcal{S}) = \sum_{m,n \in \mathbb{Z}} J_{m,n} S_{m,n} T_{-m,-n},$$

where

$$\mathbf{J} = \{J_{m,n}\} = \left\{ \wp \left[\begin{matrix} m \\ n \end{matrix} \right] \right\}, \quad \wp \left[\begin{matrix} m \\ n \end{matrix} \right] = \wp \left(\frac{m+n\tau}{N}; \tau \right). \quad (4.5)$$

Then (4.4) takes the form

$$\partial_t S_{m,n} = \sum_{k,l \in \mathbb{Z}} S_{k,l} S_{m-k,n-l} \wp \left[\begin{matrix} k \\ l \end{matrix} \right] \sin \left(\frac{\pi}{N} (ml - kn) \right). \quad (4.6)$$

It was observed in Ref. [5] that ER is a Hitchin system corresponding to the vector bundle E of rank N and degree one over the elliptic curve E_τ with the marked point $z = 0$. To prove this fact we first demonstrate that (4.6) is equivalent to the Lax equation

$$\partial_t L^{rot}(z) = [L^{rot}(z), M^{rot}(z)],$$

and then show that $L^{rot}(z)$ is the Higgs field in corresponding bundle (see Appendix A). The Lax matrices in the basis $T_{m,n}$ of $\mathfrak{gl}(N, \mathbb{C})$ are represented as

$$L^{rot} = \sum_{m,n \in \mathbb{Z}} S_{m,n} \wp \left[\begin{matrix} m \\ n \end{matrix} \right] (z) T_{m,n}, \quad \wp \left[\begin{matrix} m \\ n \end{matrix} \right] (z) = \mathbf{e} \left(\frac{nz}{N} \right) \phi \left(\frac{m+n\tau}{N}; z \right), \quad (4.7)$$

$$M^{rot} = - \sum_{m,n \in \mathbb{Z}} S_{m,n} f \left[\begin{matrix} m \\ n \end{matrix} \right] (z) T_{m,n}, \quad f \left[\begin{matrix} m \\ n \end{matrix} \right] (z) = \mathbf{e} \left(\frac{nz}{N} \right) \partial_u \phi(u; z) \big|_{u=\frac{m+n\tau}{N}}. \quad (4.8)$$

They lead to the Lax equation for the matrix elements

$$\partial_t S_{m,n} \wp \left[\begin{matrix} m \\ n \end{matrix} \right] (z) = \sum_{k,l \in \mathbb{Z}} S_{m-k,n-l} S_{kl} \wp \left[\begin{matrix} m-k \\ n-l \end{matrix} \right] (z) f \left[\begin{matrix} k \\ l \end{matrix} \right] (z) \sin \frac{\pi}{N} (nk - ml).$$

Using the Calogero functional equation (B.14) we rewrite it in the form (4.6).

The phase space \mathcal{R}^{rot} (4.1) is the result of the Hamiltonian reduction of the GL_N Higgs bundle of degree one. In this case there is no moduli degrees of freedom except the Jacobian of the determinant bundle (A.4). In fact, the determinant bundle coincides with the theta-bundle and therefore has degree one. It implies that the gauge fixing is complete and the reduced phase space is the orbit \mathcal{O} (see (A.9)). In the symplectic form (A.10) survives only the last term that coincides with (4.2). For a generic orbit $\dim \mathcal{R}^{(1)} = N(N-1)$. The transition functions can be chosen in the form (A.6). The transition functions (A.6) allow us to define the Lax operator depending only on the orbit variables. It can be checked directly that the Lax operator (4.7) is a meromorphic $\mathfrak{gl}(N, \mathbb{C})$ -valued one form on E_τ

$$\text{Res} L^{rot}|_{z=0} = \mathcal{S},$$

that satisfies the quasi-periodicity conditions with the transition functions (A.7)

$$L^{rot}(z) - Q_N L^{rot}(z+1) Q_N^{-1} = 0, \quad L^{rot}(z) - \Lambda_N L^{rot}(z+\tau) \Lambda_N^{-1} = 0.$$

It follows from the general prescription that we have $\frac{N(N-1)}{2}$ independent integrals of motion (A.20). In particular,

$$\frac{1}{2}\text{tr}(L^{\text{rot}}(z))^2 = -\frac{i\pi}{N}H^{\text{rot}} + \text{tr}\mathcal{S}^2\wp(z).$$

The equations of motion, corresponding to the higher integrals has the Lax form (A.19). The properties of $M_{s,j}(z)$ can be read of from the equation of motion (A.17) restricted to \mathcal{R}^{red}

$$M_{s,j}(z) - Q_N M_{s,j}(z+1)Q_N^{-1} = 0.$$

For $s = 0$ we have

$$M_{0,j}(z) - \Lambda_N M_{0,j}(z+\tau)\Lambda_N^{-1} = 2\pi i(L)^{j-1}(z). \quad (4.9)$$

If $s \neq 0$ then $M_{s,j}(z)$ is quasi-periodic

$$M_{s,j}(z) - \Lambda_N M_{s,j}(z+\tau)\Lambda_N^{-1} = 0,$$

and its singular part is defined by the singular part of $L_N^{j-1}z^s$

$$(M_{s,j}(z))_- = \left(L_N^{j-1}z^s\right)_-. \quad (4.10)$$

5 Elliptic rotator on \mathcal{A}_θ

1. Description of the system.

It follows from (2.10) that the non-commutative torus \mathcal{A}_θ corresponds to a special limit $N \rightarrow \infty$ of the $\text{SL}(N, \mathbb{C})$. Consider ER related to the group of NCT G_θ and assume that θ is a irrational number. We replace the inverse inertia-tensor Δ^{-1} of the hydrodynamics on the operator $\mathbf{J} : \mathcal{S} \rightarrow \psi$ acting in a diagonal way on the Fourier coefficients (3.9):

$$\mathbf{J} : s_{m,n} \rightarrow \cdot s_{m,n} = \psi_{m,n}, \quad (s_{00} = 0), \quad \wp \left[\begin{matrix} m \\ n \end{matrix} \right] = \wp((m+n\tau)\theta; \tau). \quad (5.1)$$

We consider EAT on the group G_θ with the inertia-tensor defined by \mathbf{J}^{-1} (5.1). The corresponding coadjoint orbit is

$$\mathcal{O}_{\mathcal{S}^0} = \{\mathcal{S} \in \text{sin}_\theta^* \mid \mathcal{S} = h^{-1}\mathcal{S}_0h, \quad h \in G_\theta, \quad \mathcal{S}_0 \in \text{sin}_\theta^*\} \quad (5.2)$$

equipped with the Kirillov-Kostant symplectic form

$$\omega_\theta = \int_{\mathcal{A}_\theta} \mathcal{S}^0 Dhh^{-1} \wedge Dhh^{-1}.$$

The Poisson structure on the coalgebra sin_θ^* is defined by the Moyal brackets

$$\{\mathcal{S}, \mathcal{S}'\} = \{\mathcal{S}, \mathcal{S}'\}^*.$$

Let

$$\mathcal{S} = -4\pi^2\theta^3 \sum_{m,n \in \mathbb{Z}} s_{m,n} T_{-m,-n} \in \text{sin}_\theta^*, \quad (s_{0,0} = 0),$$

and

$$\mathbf{J}(\mathcal{S}) = \sum_{m,n \in \mathbb{Z}} s_{-m,-n} \wp \left[\begin{matrix} m \\ n \end{matrix} \right] T_{m,n} \in \text{sin}_\theta.$$

The Hamiltonian is determined by the integral over \mathcal{A}_θ (2.8)

$$H_\theta = -\frac{1}{2} \int_{\mathcal{A}_\theta} \mathbf{S} \mathbf{J}(\mathcal{S}) = -\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \wp \begin{bmatrix} m \\ n \end{bmatrix} s_{m,n} s_{-m,-n}. \quad (5.3)$$

We define the phase space below assuming now that the Hamiltonian H_θ is finite. The equation of motion has the standard form of the Moyal brackets, or the commutator in $\text{GL}(\infty)$

$$\partial_t \mathcal{S} = \{\mathcal{S}, \mathbf{J}(\mathcal{S})\}^* = [\mathcal{S}, \mathbf{J}(\mathcal{S})]. \quad (5.4)$$

In the Fourier components it takes the form

$$\partial_t s_{m,n} = \frac{1}{\pi\theta} \sum_{j,k \in \mathbb{Z}} s_{jk} s_{m-j,n-k} \times \wp \begin{bmatrix} j \\ k \end{bmatrix} \sin(\pi\theta(jn - km)). \quad (5.5)$$

2. Integrability of elliptic rotator on \mathcal{A}_θ .

We will prove that the Hamiltonian system of ER (5.4), (5.5) has an infinite set of involutive integrals of motion in addition to the Casimirs (3.6). It will follow from the Lax form

$$\partial_t L_\theta = [L_\theta, M_\theta] \quad (5.6)$$

of the equations (5.4), (5.5). The Lax operators are similar to the corresponding Lax matrices for the elliptic rotator (4.7), (4.8)

$$L_\theta = \sum_{mn \in \mathbb{Z}} s_{m,n} \wp \begin{bmatrix} m \\ n \end{bmatrix} (z) T_{m,n}, \quad M_\theta = - \sum_{mn \in \mathbb{Z}} s_{m,n} f \begin{bmatrix} m \\ n \end{bmatrix} (z) T_{m,n}, \quad (5.7)$$

where

$$\wp \begin{bmatrix} m \\ n \end{bmatrix} (z) = \mathbf{e}(n\theta z) \phi((m + n\tau)\theta, z), \quad (5.8)$$

$$f \begin{bmatrix} m \\ n \end{bmatrix} (z) = \mathbf{e}(n\theta z) \partial_u \phi(u, z)|_{u=(m+n\tau)\theta}, \quad (5.9)$$

and $\phi(u, z)$ is defined in (B.8). The equivalence of (5.6) and (5.5) follows from the Calogero functional equation (B.14).

Consider the holomorphic vector bundle E of infinite rank over E_τ with the structure group G_θ . We assume that it is similar to the $\text{GL}(N, \mathbb{C})$ bundle of degree one (see Appendix A), where $\text{GL}(N, \mathbb{C})$ is replaced by G_θ . It means that the transition functions g_α , $\alpha = 1, 2$ have the form

$$g_1 = Q, \quad g_2 = \tilde{\Lambda} = \mathbf{e}((-\frac{1}{2}\tau + z)\theta)\Lambda.$$

The Higgs bundle is $(T^*E, \mathcal{O}_{S^0})$, where the coadjoint orbit \mathcal{O}_{S^0} is defined by (5.2). The cotangent bundle T^*E is described by the Higgs field $\Phi = f^{-1}(z)L_\theta f(z)$. The Lax operator L_θ satisfies the moment constraint equation

$$\bar{\partial} L_\theta = 0, \quad \text{Res } L_\theta|_{z=0} = \mathcal{S}, \quad (5.10)$$

$$L_\theta(z+1) = QL_\theta(z)Q^{-1}, \quad L_\theta(z+\tau) = \Lambda L_\theta(z)\Lambda^{-1}. \quad (5.11)$$

The reduced phase space is described by solutions of (5.10), (5.11) such that

$$I_{s,j} = \int_{E_\tau} \int_{\mathcal{A}_\theta} (L_\theta)^j \mu_{s,j} < \infty, \quad (s \leq j, j \in \mathbb{N}), \quad (5.12)$$

and $\mu_{s,j}$ are defined by (A.14) and (B.22). The integrals $I_{s,j}^{er}$ can be extracted from the expansion over the basis of the elliptic functions (B.20)

$$\int_{\mathcal{A}_\theta} (L_\theta)^j(z) = I_{0,j} + \sum_{r=2}^j I_{r,j} \wp^{(r-2)}(z), \quad (j = 2, \dots).$$

In particular,

$$\int_{\mathcal{A}_\theta} (L_\theta)^2(z) = I_{0,2} + \wp(z) \int_{\mathcal{A}_\theta} \mathcal{S}^2, \quad I_{0,2} = 2\pi^2 \theta^2 H_\theta.$$

Note that

$$I_{j,j} \sim C_j = \int_{\mathcal{A}_\theta} \mathcal{S}^j$$

are the Casimirs (3.6).

Consider, for example, the integrals, that have the third order in the field \mathcal{S} . It follows from (B.18) that in terms of the Fourier modes $\mathcal{S} = \{s_{m,n}\}$ the integrals take the form

$$I_{2,3} = \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^3 s_{m_j n_j} \left(\zeta \left[\begin{smallmatrix} m_1 \\ n_1 \end{smallmatrix} \right] + \zeta \left[\begin{smallmatrix} m_2 \\ n_2 \end{smallmatrix} \right] + \zeta \left[\begin{smallmatrix} m_3 \\ n_3 \end{smallmatrix} \right] \right), \quad (5.13)$$

$$\left(\zeta \left[\begin{smallmatrix} m \\ n \end{smallmatrix} \right] = \zeta((m + n\tau)\theta; \tau) \right),$$

$$I_{0,3} = \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^3 s_{m_j n_j} \times \left(-\frac{1}{2} \wp' \left[\begin{smallmatrix} m_3 \\ n_3 \end{smallmatrix} \right] - \wp \left[\begin{smallmatrix} m_3 \\ n_3 \end{smallmatrix} \right] \left[\zeta \left[\begin{smallmatrix} m_1 \\ n_1 \end{smallmatrix} \right] + \zeta \left[\begin{smallmatrix} m_2 \\ n_2 \end{smallmatrix} \right] + \zeta \left[\begin{smallmatrix} m_3 \\ n_3 \end{smallmatrix} \right] \right] \right). \quad (5.14)$$

The functionals (5.12) play the role of Hamiltonians for the infinite hierarchy on the phase space $\mathcal{O}_{\mathcal{S}^0}$ (5.2)

$$\partial_{s,j} \mathcal{S} = \{\nabla I_{s,j}, \mathcal{S}\}^* \quad (\partial_{s,j} = \partial_{t_{s,j}}, \quad \nabla I_{s,j}^{er} = \frac{\delta I_{s,j}^{er}}{\delta \mathcal{S}}). \quad (5.15)$$

It contains (5.4), (5.5) for $I_{0,2}$.

3. Classical limit.

In the classical limit \mathcal{S} becomes a function on T^2

$$\mathcal{S} = \sum_{m,n \in \mathbb{Z}} s_{m,n} \exp(2\pi i(-mx - ny)).$$

In our case the classical limit essentially is the same as the rational limit of the basic spectral curve E_τ . Replace for a moment the half periods $(\frac{1}{2}, \frac{\tau}{2})$ of E_τ on ω_1, ω_2 . The rational limit means that $\omega_1, \omega_2 \rightarrow \infty$. The Weierstrass function degenerates as

$$\wp(u) \rightarrow \frac{1}{u^2}.$$

Consider the double limit $\theta \rightarrow 0$, $\omega_1, \omega_2 \rightarrow \infty$ such that $\lim \omega_1 \theta = 1$, $\lim \omega_2 \theta = \tau$. Then

$$\wp \left[\begin{smallmatrix} m \\ n \end{smallmatrix} \right] \rightarrow \frac{1}{(m + n\tau)^2}.$$

The quadratic Hamiltonian in the double limit takes the form

$$H = -\frac{1}{2} \int_{\mathcal{A}_\theta} \psi(\bar{\partial})^2 \psi = -\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \frac{s_{m,n} s_{m,n}}{(m + n\tau)^2}, \quad (5.16)$$

where $\bar{\partial} = \frac{1}{2\pi i}(\partial_x + \tau \partial_y)$. The operator $\bar{\partial}^2$ plays the role of the inertia-tensor. It replaces the Laplace operator $\Delta \sim \partial \bar{\partial}$ of the standard hydrodynamics. We call this system the *modified hydrodynamics*.

In the classical limit the Lax equation assumes the form

$$\partial_t L(x_1, x_2; z) = \{L(x_1, x_2; z), M(x_1, x_2)\},$$

where

$$L(x, y; z) = \bar{\partial}^{-1} \mathcal{S}(x, y) + \frac{1}{z} \mathcal{S}(x, y), \quad (5.17)$$

and

$$M(x, y) = -\bar{\partial}^{-2} \mathbf{S}(x, y). \quad (5.18)$$

The integrals of motion (5.12) survive in this limit. We already pointed the form of the Hamiltonian H (5.16). The third order integrals take the forms

$$\begin{aligned} I_{2,3} &= \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^3 s_{m_j, n_j} \sum_j \frac{1}{m_j + n_j \tau}, \\ I_{0,3} &= \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^3 s_{m_j, n_j} \times \\ &\quad \left(-\frac{1}{(m_3 + n_3 \tau)^3} - \frac{1}{(m_3 + n_3 \tau)^2} \left[\frac{1}{m_1 + n_1 \tau} + \frac{1}{m_2 + n_2 \tau} + \frac{1}{m_3 + n_3 \tau} \right] \right). \end{aligned}$$

4. Reduction to the loop algebra

Let θ be a rational number $\theta = p/N$. As it was explained in Section 2.3 we can pass to the factor-algebra $L(\mathfrak{gl}(N, \mathbb{C}))$ or its central extension $\hat{L}(\mathfrak{gl}(N, \mathbb{C}))$. In the first case we have a family of non-interacting ER parameterized by $w \in S^1$. If the central charge nonzero the situation is drastically changed [2, 27]. The Lax operator is no longer a one-form, but a connection on S^1

$$\partial_w + L(z, w), \quad (w \in S^1). \quad (5.19)$$

The integrals of motion can be calculated from the expansion of the trace of the monodromy matrix for the linear system

$$(\partial_w + L(z, w))\Psi(z, w) = 0.$$

They define the hierarchy of the ER on the coadjoint orbits of $\hat{L}(\mathfrak{GL}(N, \mathbb{C}))$. For $N = 2$ this top is just the Landau-Lifshitz equation

$$\partial_t \mathcal{S} = \frac{1}{2} [\mathcal{S}, J(\mathcal{S})] + \frac{1}{2} [\mathcal{S}, \partial_{ww} \mathcal{S}]. \quad (5.20)$$

Here $\mathcal{S} \in L^*(\mathfrak{sl}(2, \mathbb{C}))$.

6 Elliptic Calogero-Moser system on \mathcal{A}_θ

1. $\mathrm{SL}(N, \mathbb{C})$ -Elliptic Calogero-Moser system (CM_N)

The elliptic CM_N system was first introduced in the quantum version [24] and then in the classical [25]. We consider its generalization - CM_N system with spin. The elliptic CM_N corresponds to the trivial Higgs bundle over the elliptic curve E_τ [26]. Its phase space is

$$\mathcal{R}^{CM_N} = \{\mathbb{C}^{2N}, \tilde{\mathcal{O}}\}, \quad (6.1)$$

where $\tilde{\mathcal{O}} = \mathcal{O} // D$ is the symplectic quotient of the coadjoint orbit of $\mathrm{SL}(N, \mathbb{C})$

$$\mathcal{O} = \{p \in \mathfrak{sl}(N, \mathbb{C}) \mid p = h^{-1}p^0h, \ h \in \mathrm{SL}(N, \mathbb{C}), p^0 \in D\} \quad (6.2)$$

with respect to the action of the diagonal subgroup D of $\mathrm{SL}(N, \mathbb{C})$. The moment constraint imply that the diagonal matrix elements of the orbit vanish $p_{jj} = 0$. The space \mathcal{R}^{CM_N} has the same dimension $\dim \mathcal{R}_{(0)}^{red} = N(N-1)$ as for the elliptic rotator.

The Poisson structure has the form

$$\{v_j, u_k\} = \delta_{j,k}, \quad \{p_{k,l}, p_{j,n}\} = \delta_{j,l}p_{k,n} - \delta_{n,k}p_{j,l}, \quad (6.3)$$

where $\vec{v} = (v_1, \dots, v_N)$, $\vec{u} = (u_1, \dots, u_N)$ are canonical coordinates on \mathbb{C}^{2N} .

The Hamiltonian, that has the second order with respect to the momenta \mathbf{v} , has the form

$$H_2^{CM_N} = \frac{1}{2} \sum_{j=1}^N v_j^2 + \sum_{j>k} p_{jk} p_{kj} \wp(u_j - u_k; \tau). \quad (6.4)$$

It describes the interaction of N particles with complex coordinates u_1, \dots, u_N on the elliptic curve E_τ (B.1). The pair-wise potential is defined by the Weierstrass function. The spin degrees of freedom p_{jk} looks like EAT with the inertia-tensor determined by $\wp(u_j - u_k; \tau)$, but the corresponding phase subspace in contrast with standard EAT is the symplectic quotient $\mathcal{O} // D$.

The equation of motion with respect to H_2^{CM} has the Lax form $\partial_t L^{CM_N} = [L^{CM_N}, M^{CM_N}]$ with

$$L^{CM_N} = P + X, \text{ where } P = \mathrm{diag}(v_1, \dots, v_N), \quad X_{jk} = p_{jk} \phi(u_j - u_k, z), \quad (6.5)$$

(ϕ is defined by (B.8)) and

$$M^{CM_N} = -D + Y, \text{ where } D = \mathrm{diag}(Z_1, \dots, Z_N), \quad Y_{jk} = y(u_j - u_k, z), \quad (6.6)$$

$$Z_j = \sum_{k \neq j} \wp(u_j - u_k), \quad y(u, z) = \frac{\partial \phi(u, z)}{\partial u}.$$

The equivalence of the Lax equation and the equations of motion is based again on (B.14) and (B.12).

We use relations from Appendix A to derive the elliptic CM_N system and its Lax representation via the Hitchin construction [26]. If $d = \deg(E_N^{st}) = 0$, then the transition functions g_α can be gauge transformed to the constant matrices (A.5). The Lax operator is a meromorphic matrix-valued one-form. Its quasi-periodicity properties are defined by the transition functions (A.5)

$$L^{CM_N}(z+1) = L^{CM_N}(z), \quad L^{CM_N}(z+\tau) = \exp(\vec{u}) L^{CM_N}(z) \exp(-\vec{u}).$$

It has a simple pole at $z = 0$ such that

$$\mathrm{Res}_{z=0}(L^{CM_N}(z)) = L_{-1}^{CM_N} = p \in \tilde{\mathcal{O}}. \quad (6.7)$$

The integrals of motion $I_{s,j}$ (A.20) produce CM_N hierarchy

$$\partial_{s,j} L^{CM_N} = [L^{CM_N}, M_{s,j}^{CM_N}]. \quad (6.8)$$

The properties of $M_{s,j}^{CM_N}$ can be extracted from the equations of motion (A.17)

$$\begin{aligned} M_{s,j}^{CM_N}(z+1) &= M_{s,j}^{CM_N}(z), \\ M_{0,j}^{CM_N}(z) - \exp(\vec{u}) M_{0,j}^{CM_N}(z+\tau) \exp(-\vec{u}) &= 2\pi i (L^{CM_N})^{j-1} - \partial_{0,j} \vec{u}. \end{aligned} \quad (6.9)$$

For $s \neq 0$ we have

$$M_{s,j}^{CM_N}(z) - \exp(\vec{u}) M_{s,j}^{CM_N}(z+\tau) \exp(-\vec{u}) = -\partial_{s,j} \vec{u},$$

and the singular part of $M_{s,j}^{CM_N}(z)$ has the form

$$(M_{s,j}^{CM_N}(z))_- = (L^{CM_N}(z))^{j-1} z^s_-.$$

In particular, $I_{0,2} = H_2^{CM}$ and $M_{0,2}^{CM_N} = M^{CM_N}$ (6.6).

Let $f(z)$ be the gauge transformation that diagonalized g_2 . It is defined up to the conjugation by a constant diagonal matrix. This remnant gauge freedom is responsible for the symplectic reduction of the orbit $\tilde{\mathcal{O}} = \mathcal{O}/\mathcal{D}$.

2. Equilibrium configuration

We prove now that the following configuration of particles and spins is an equilibrium set with respect to the Hamiltonian $H_2^{CM_N}$ (6.4). Consider $N = n^2$ particles and the orbit variables enumerated by the pair of integer numbers $a, b = 1, \dots, n$

$$p_{a,b,c,d} = \nu, \quad v_{a,b} = 0, \quad u_{a,b} = \frac{a+b\tau}{N} \quad a, b = \overline{1, n}. \quad (6.10)$$

From the identity

$$\wp(Nz|\tau) = \frac{1}{N^2} \left[\wp(z|\tau) + \sum_{j=1}^N \left(\sum_{k=1}^{N-1} \wp\left(z + \frac{j+k\tau}{N}|\tau\right) \right) + \wp\left(z + \frac{j}{N}|\tau\right) \right]. \quad (6.11)$$

one obtains

$$\sum_{j=1}^N \left(\sum_{k=1}^{N-1} \wp\left(\frac{j+k\tau}{N}|\tau\right) \right) + \wp\left(\frac{j}{N}|\tau\right) = 0. \quad (6.12)$$

It follows from (B.13) and (6.11) that

$$\wp'\left(\frac{j+k\tau}{N}|\tau\right) + \sum_{m \neq j, n \neq k} \wp'\left(\frac{j+k\tau}{N} - \frac{m+n\tau}{N}|\tau\right) = 0. \quad (6.13)$$

Then (6.13) implies that (6.10) is the equilibrium set in \mathcal{R}^{CM_N} with respect to $H_2^{CM_N}$ (6.4). Moreover, (6.12) means that the Hamiltonian (6.4) vanishes at this point

$$H_2^{CM} = 0. \quad (6.14)$$

Note, that configuration (6.10) is preserved by the action of the higher integrals $I_{s,k}$.

3. Symplectic Hecke correspondence

There exists a canonical transformation (*Symplectic Hecke correspondence*) that defines the pass from CM_N model related to the Higgs bundle of degree zero to ER on $\text{GL}(N, \mathbb{C})$ related to Higgs bundle of degree one [2]. It is a singular gauge transformation Ξ with a special form of its kernel. An eigen-vector of the residue $L_1^{CM_N}$ (6.7) is annihilated by the kernel. Then this gauge transform

$$L^{er} = \Xi^{-1} L_{2D}^{CM} \Xi. \quad (6.15)$$

preserves the order of the pole. The matrix Ξ has the following form. Let $p^0 = \text{diag}(p_1, \dots, p_N)$ be the diagonal matrix defining the coadjoint orbit (6.2) in the elliptic CM_N system. Then $\Xi = \Xi(p_l)$ depends on a choice of the eigen-value p_l . Consider the following $(N \times N)$ - matrix $\tilde{\Xi}(z, u_1, \dots, u_N; \tau)$:

$$\tilde{\Xi}_{ij}(z, \mathbf{u}; \tau) = \theta \left[\begin{array}{c} \frac{i}{N} - \frac{1}{2} \\ \frac{N}{2} \end{array} \right] (z - Nu_j, N\tau),$$

where $\theta \left[\begin{array}{c} a \\ b \end{array} \right] (z, \tau)$ is the theta function with characteristics (B.24). Then

$$\Xi(z, \mathbf{u}, p_l; \tau) = \tilde{\Xi}(z) \times \text{diag} \left(\frac{(-1)^l}{p_l} \prod_{j < k; j, k \neq l} \vartheta(u_k - u_j, \tau) \right). \quad (6.16)$$

Consider the case $N = 2$. The phase space has dimension two, since the orbit variables (6.2) are gauged away. Let ν^2 be the value of the Casimir of the orbit. Then the transformation takes the form

$$\begin{cases} S_1 = -v \frac{\theta_{10}(0)}{\vartheta'(0)} \frac{\theta_{10}(2u)}{\vartheta(2u)} - \nu \frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)} \frac{\theta_{00}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}, \\ S_2 = -v \frac{\theta_{00}(0)}{\sqrt{-1}\vartheta'(0)} \frac{\theta_{00}(2u)}{\vartheta(2u)} - \nu \frac{\theta_{00}^2(0)}{\sqrt{-1}\theta_{10}(0)\theta_{01}(0)} \frac{\theta_{10}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}, \\ S_3 = -v \frac{\theta_{01}(0)}{\vartheta'(0)} \frac{\theta_{01}(2u)}{\vartheta(2u)} - \nu \frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)} \frac{\theta_{00}(2u)\theta_{10}(2u)}{\vartheta^2(2u)}, \end{cases} \quad (6.17)$$

where $\nu^2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$.

4. Elliptic CM system on \mathcal{A}_θ

Consider the limit $N \rightarrow \infty$ of CM_N system CM_∞ corresponding to \mathcal{A}_θ . We identify the coordinates of infinite number of particles in E_τ with the diagonal matrix in GL_∞

$$\vec{u} = \text{diag}(\dots, u_{-N}, \dots, u_{-1}, u_0, u_1, \dots, u_N, \dots),$$

and let

$$\vec{v} = \text{diag}(\dots, v_{-N}, \dots, v_{-1}, v_0, v_1, \dots, v_N, \dots).$$

be their momenta.

The Hamiltonian of CM_∞ has the form

$$H^{CM_\infty} = \frac{1}{2}(\vec{v}, \vec{v}) + \sum_{j < k, j, k \in \mathbb{Z}} p_{jk} p_{kj} \wp(u_j - u_k; \tau), \quad (6.18)$$

where the orbit elements p_{jk} is written in the terms of the generators E_{jk} . In spite of the infinite number of the particles on the torus E_τ the Hamiltonian H^{CM_∞} remains finite around the equilibrium configuration (6.10), (6.14).

The phase space \mathcal{R}^{CM_∞} of CM_∞ has the similar form as in the finite-dimensional case (6.1)

$$\mathcal{R}^{CM_\infty} = \{\mathbb{C}^\infty \oplus \mathbb{C}^\infty; \tilde{\mathcal{O}}_\infty\}.$$

Here $\tilde{\mathcal{O}}_\infty = \mathcal{O}_\infty // D$ is the symplectic quotient of the coadjoint orbit with respect to the Cartan subgroup $D \subset SIN_\theta$ generated by $T_{m,0}$, $m \in \mathbb{Z}$. The coadjoint orbit $\mathcal{O}_\infty \subset \sin_\theta^*$ of $SIN_\theta \subset GL_\infty$ is

$$\mathcal{O}_\infty = \{p \in \sin_\theta^* \mid p = h^{-1}p^0h, h \in SIN_\theta\}.$$

We assume that $\vec{v} \in \mathbb{C}^\infty$ satisfies (6.23). There are additional restrictions coming from the finiteness of the integrals (6.33) defined below.

In terms of coordinates on \mathfrak{gl}_∞^* the Poisson brackets on are given by the similar formulae as (6.3)

$$\{v_j, u_k\} = \delta_{j,k}, \quad (6.19)$$

$$\{p_{k,l}, p_{j,n}\} = \delta_{j,l}p_{k,n} - \delta_{n,k}p_{j,l}. \quad (6.20)$$

We express the coordinates of the particles in terms of the coordinates on \mathcal{A}_θ
 $\vec{u} = \sum_{j \neq 0} \tilde{u}_j T_{j,0}$

$$u_j = \frac{i}{2\pi\theta} \sum_{k \in \mathbb{Z}} \tilde{u}_k \mathbf{e}(jk\theta), \quad \tilde{u}_k \in \mathfrak{S}. \quad (6.21)$$

Evidently, u_j is represented by the convergent series. Similarly,

$$v_j = -2\pi i \theta^2 \sum_{k \in \mathbb{Z}} \tilde{v}_k \mathbf{e}(jk\theta). \quad (6.22)$$

where $\tilde{v}_k \in \mathfrak{S}'$ (2.4) and we assume that

$$\sum_{k \in \mathbb{Z}} (v_k)^j < \infty, \quad j = 2, 3, \dots \quad (6.23)$$

Consider the generating functions

$$\mathbf{u}(x) = \frac{i}{2\pi\theta} \sum_{m \in \mathbb{Z}} \tilde{u}_m \mathbf{e}(x)^m, \quad (6.24)$$

and

$$\mathbf{v}(x) = -2\pi i \theta^2 \sum_{m \in \mathbb{Z}} \tilde{v}_m \mathbf{e}(x)^{-m}, \quad (6.25)$$

where we identify $\mathbf{e}(x)$ with the generator U_1 . In terms of coordinates $(\tilde{v}_m, \tilde{u}_l)$ the Poisson brackets takes the form

$$\{\tilde{v}_m, \tilde{u}_n\} = \delta_{m,n}, \quad (6.26)$$

or

$$\{\mathbf{v}(x), \mathbf{u}(x')\} = \delta(x - x'), \quad (6.27)$$

where $\delta(x) = \sum_{m \in \mathbb{Z}} \mathbf{e}(m\theta x)$.

Define the orbit variables in the basis $T_{m,n}$:

$$\mathcal{S}(x, y) = -2\pi i \theta^2 \sum_{m,n} \mathbf{e}\left(\frac{mn\theta}{2}\right) s_{m,n} \mathbf{e}(x)^{-m} * \mathbf{e}(y)^{-n}, \quad (U_1 \sim \mathbf{e}(x), U_2 \sim \mathbf{e}(y)). \quad (6.28)$$

It follows from (2.11) that in terms of the coordinates on the NCT $s_{m,n}$ the orbit variables are expanded as

$$p_{j,j+n} = -2\pi i \theta^2 \sum_{m \in \mathbb{Z}} \mathbf{e}\left(m\theta\left(\frac{n}{2} - j\right)\right) s_{m,n}. \quad (6.29)$$

Since $p_{j,j} = 0$, $s_{m,0} = 0$ and $\mathcal{S}(x, 0) \equiv 0$. The brackets (6.20) takes the form

$$\{s_{m,n}, s_{m',n'}\} = \frac{1}{\pi\theta} \sin(\pi\theta(mn' - m'n)) s_{m+m', n+n'}. \quad (6.30)$$

The Hamiltonian (6.18) can be rewritten in terms of the NCT variables. Using (6.21) and (6.24) we find

$$\wp(u_j - u_{j+n}; \tau) = \wp(\mathbf{u}(\theta j) - \mathbf{u}(\theta(j+n)); \tau).$$

Similarly to (6.29) we define the coefficients $r_{m,n}$ as

$$\wp(\mathbf{u}(\theta j) - \mathbf{u}(\theta(j+n))) = \sum_{m \in \mathbb{Z}} \mathbf{e}\left(m\theta\left(\frac{n}{2} - j\right)\right) r_{m,n}.$$

and the corresponding function on \mathcal{A}_θ

$$\mathcal{P}(x, y) = \sum_{m,n} \mathbf{e}\left(\frac{mn\theta}{2}\right) r_{m,n} \mathbf{e}(x)^{-m} * \mathbf{e}(y)^{-n}.$$

Thus,

$$\wp(u_j - u_{j+n}; \tau) p_{j,j+n} = -2\pi\theta^2 \sum_m \mathbf{e}\left(m\theta\left(\frac{n}{2} + j\right)\right) \sum_k s_{k-m,n} r_{k,n}.$$

It allows us to put in the correspondence to the product $\wp(u_j - u_{j+n}; \tau) p_{j,j+n}$ the "convolution"

$$(\mathcal{P} \odot \mathcal{S})(x, y) := -2\pi\theta^2 \sum_{m,n} \mathbf{e}\left(\theta\frac{mn}{2}\right) \left(\sum_k r_{k,n} s_{k-m,n} \right) \mathbf{e}(x)^{-m} * \mathbf{e}(y)^{-n}.$$

Along with (6.25) and (6.28) it leads to the following expression for the Hamiltonian (6.18)

$$H^{CM_\infty} = \frac{1}{2} \int_{\mathcal{A}_\theta} \mathbf{v}(x)^2 dx + \int_{\mathcal{A}_\theta} (\mathcal{P} \odot \mathcal{S})(x, y) * \mathcal{S}(x, y).$$

The CM_∞ comes from the trivial infinite rank Higgs bundle over E_τ with transition functions $g(z) \in SIN_\theta$. The whole procedure is similar to finite-dimensional case. In particular,

$$L^{CM_\infty} = P + X.$$

Here

$$P = \text{diag}(\dots, v_{-N}, \dots, v_{-1}, v_0, v_1, \dots, v_N, \dots), \quad (6.31)$$

$$X_{jk} = p_{jk} \phi(u_j - u_k, z) \quad p_{jk} \in \tilde{\mathcal{O}}_\infty. \quad (6.32)$$

Define the coefficients $\tilde{\phi}_{m,n}$ by the expansion

$$\phi(u_j - u_{j+n}; z) \equiv \phi(\mathbf{u}(\theta j) - \mathbf{u}(\theta(j+n)); z) = \sum_m \mathbf{e}(m\theta(\frac{n}{2} - j)) \tilde{\phi}_{m,n}(\mathbf{u}; z).$$

and construct the generating function

$$\mathcal{F}(\mathbf{u}, x, y; z) = \frac{i}{2\pi\theta} \sum_{m,n} \tilde{\phi}_{m,n}(\mathbf{u}; z) \mathbf{e}\left(\frac{mn\theta}{2}\right) \mathbf{e}(x)^{-m} * \mathbf{e}(y)^{-n}.$$

In the terms of the NCT $L^{CM\infty}$ has the form

$$L^{CM\infty}(x, y) = \mathbf{v}(x) + (\mathcal{S} \odot \mathcal{F}(\mathbf{u}, x, y; z))(x, y),$$

where \mathbf{v} and \mathcal{S} are defined by (6.25) and (6.28).

We have the infinite set of the integrals of motion

$$I_{s,j} = \int_{E_\tau} \int_{\mathcal{A}_\theta} (L^{CM\infty})^j \mu_{s,j}, \quad (6.33)$$

and we assume that they are finite $I_{s,j} < \infty$. In particular,

$$\int_{\mathcal{A}_\theta} (L^{CM\infty})^2(z) = I_{0,2} + \wp(z) I_{2,2}, \quad I_{2,2} = \int_{\mathcal{A}_\theta} \mathcal{S}^2, \quad H^{CM\infty} = \frac{1}{2} I_{0,2}.$$

The integrals (5.12) give rise to the hierarchy of the commuting flows $\partial_{s,j} \sim \{I_{s,j}, \}$.

5. Reduction to the loop algebra.

For a rational number $\theta = p/N$ one can pass to $\hat{L}(\mathfrak{gl}(N, \mathbb{C}))$. The Lax operator being a one-form on S^1 (5.19) acquires a form [2, 27]

$$L^{CM} = -\frac{\delta_{ij}}{2\pi\sqrt{-1}} \left(\frac{v_i}{2} + \sum_{\alpha} p_{ii}^{\alpha} E_1(z - w_{\alpha}) \right) - \frac{1 - \delta_{ij}}{2\pi\sqrt{-1}} \sum_{\alpha} p_{ij}^{\alpha} \phi(u_{ij}, z - w_{\alpha}).$$

The integrals of motion can be calculated from the expansion of the trace of the monodromy matrix for the linear system

$$(\partial_w + L(z, w))\Psi(z, w) = 0.$$

They define the hierarchy of the elliptic CM field theory. For $N = 2$ the first non-trivial integral has the form

$$H = \oint \frac{dw}{w} \left(-\frac{v^2}{16\pi^2} \left(1 - \frac{u_w^2}{h} \right) + (3u_w^2 - h) \wp(2u) - \frac{u_{ww}^2}{4\nu^2} \right), \quad (6.34)$$

where h is a Casimir corresponding to the co-adjoint orbit of $\hat{L}(\mathfrak{GL}(N, \mathbb{C}))$ and $\nu^2 = h - u_w^2$. For an arbitrary N the quadratic Hamiltonians of the type $I_{0,2}$ were calculated in Ref. [27].

Let L^{LL} be the Lax operator for the Landau-Lifshitz equation and L_{2D}^{CM} the Lax operator corresponding to (6.34). Then (see(6.15))

$$L^{LL} = \Xi^{-1} \partial_w \Xi + \Xi^{-1} L_{2D}^{CM} \Xi,$$

where Ξ is defined by (6.16) for $N = 2$. The explicit relations between the phase space variables are given by (6.17).

7 Conclusion

There are four related subjects that are not covered here.

- We have not considered here the classical limit of the CM_{∞} model. One can try to describe it independently as the Hitchin system with the structure group $\text{SDiff}(T^2)$.

- It can be expected that the symplectic Hecke correspondence survives in the limit $N \rightarrow \infty$. It would imply that CM system on the non-commutative torus \mathcal{A}_θ and ER on \mathcal{A}_θ are symplectomorphic. It means in particular that the former system is not far from the non-commutative modification of the 2D hydrodynamics. The symplectic Hecke correspondence

just boil the particles degrees of freedom to the orbit variables. It can be suggested that the correspondence survives in the classical limit.

- It will be interesting to define the both systems on the central extended algebra $\widehat{\sin_\theta}$ (2.15). The central charge produces the additional dimension and the corresponding systems cover the CM field theory and the Landau-Lifshitz model.

- Two different tori are incorporated in our construction - the gauge NCT \mathcal{A}_θ and the basic spectral curve E_τ . In the classical limit they become dual. It seems natural to replace E_τ on another NCT $\mathcal{A}_{\theta'}$. In a general setting it means a generalization on the Higgs bundles over the non-commutative base. The categories of holomorphic vector bundles on the non-commutative torus were constructed in the recent paper [28]. One attempt in this direction was done in Ref. [29].

8 Acknowledgments

The work was supported by the grant NSh-1999-2003.2 of the scientific schools and RFBR-03-02-17554.

9 Appendix

9.1 Appendix A. Hitchin systems on an elliptic curve.

Let E_N^{st} be a rank N stable holomorphic vector bundle over the elliptic curve E_τ (B.1). It can be described by the holomorphic $\mathrm{GL}(N, \mathbb{C})$ -valued transition functions

$$g_1(z) : z \rightarrow z + 1, \quad g_2(z) : z \rightarrow z + \tau, \quad (\text{A.1})$$

$$g_\alpha \in \Omega^0(\mathcal{U}_\alpha, \mathrm{Aut}^* E_N) \quad \alpha = 1, 2,$$

(\mathcal{U}_1 is a neighborhood of $[0, \tau]$, \mathcal{U}_2 is a neighborhood of $[0, 1]$).

They satisfy the cocycle conditions

$$g_1(z)g_2(z+1)g_1^{-1}(z+1+\tau)g_2^{-1}(z+\tau) = Id.$$

Define the action of the gauge group $\mathcal{G}_N = \{f(z)\}$ as

$$g_1(z) \rightarrow f(z)g_1(z)f^{-1}(z+1), \quad g_2(z) \rightarrow f(z)g_2(z)f^{-1}(z+\tau). \quad (\text{A.2})$$

The moduli space of the stable holomorphic bundles $\mathcal{M}_N(E_\tau)$ is defined as the quotient

$$\mathcal{M}_N(E_\tau) = \mathcal{G}_N \backslash E_N. \quad (\text{A.3})$$

The space $\mathcal{M}_N(E_\tau)$ is a disjoint union of the components labeled by the corresponding degrees $d = c_1(\det E_N) : \mathcal{M}_N(E_\tau) = \bigsqcup \mathcal{M}_N^{(d)}$. The tangent space to $\mathcal{M}(E_\tau)$ is isomorphic to $h^1(E_\tau, \mathrm{End} E_N^{st})$. Its dimension can be extracted from the Riemann-Roch theorem

$$\dim h^1(E_\tau, \mathrm{End} E_N) = \dim h^0(E_\tau, \mathrm{End} E_N).$$

As a result we have

$$\dim \mathcal{M}_N^{(d)} = g.c.d.(N, d) \quad (\text{A.4})$$

The generic stable bundles can be transformed by (A.2) to the constant diagonal form. For the trivial bundles ($d = 0$)

$$g_1^{(0)} = Id, \quad g_2^{(0)} = f^{-1}(z) \text{diag} \exp \mathbf{u} f(z). \quad (\text{A.5})$$

For $d = 1$ the transition functions can be chosen in the form

$$g_1^{(1)} = f^{-1}(z) Q_N f(z), \quad g_2^{(1)} = f^{-1}(z) \tilde{\Lambda}_N f(z), \quad \tilde{\Lambda}_N = \mathbf{e}(-\frac{\frac{1}{2}\tau + z}{N}) \Lambda_N, \quad (\text{A.6})$$

where

$$Q_N = \text{diag}(1, \mathbf{e}(\frac{1}{N}), \dots, \mathbf{e}(\frac{1}{N-1})), \quad \Lambda_N = \sum_{j=1, N, (\text{mod } N)} E_{j, j+1}. \quad (\text{A.7})$$

Consider the cotangent bundle $T^*E_N^{st}$. We choose them in the following form.

$$\eta_\alpha \in \Omega^{(1,0)}(\mathcal{U}_\alpha, \text{End}^* E_N^{st}) \quad \alpha = 1, 2,$$

(\mathcal{U}_1 is a neighborhood of $[0, \tau]$, \mathcal{U}_2 is a neighborhood of $[0, 1]$).

The bundle T^*E_N is called the *Higgs bundle* over E_τ .

We attribute to the marked point $z = 0$ a coadjoint orbit of $\text{SL}(N, \mathbb{C})$

$$\mathcal{O} = \{p \in \mathfrak{sl}(N, \mathbb{C}) \mid p = h^{-1} p^0 h, \quad h \in \text{SL}(N, \mathbb{C}), \quad p^0 \in \mathfrak{sl}(N, \mathbb{C})\}. \quad (\text{A.8})$$

The unreduced phase space is the pair

$$\mathcal{R} = (T^*E_N^{st}, \mathcal{O}) \quad (\text{A.9})$$

with the symplectic form

$$\omega = \sum_{\alpha=1,2} \oint_{\gamma_\alpha} \text{tr}(D(g_\alpha^{-1} \eta_\alpha) \wedge Dg_\alpha) + \text{tr}(D(h^{-1} p^0) \wedge Dh). \quad (\text{A.10})$$

Here the integrals \oint is taken over contours $\gamma_1 \sim [0, \tau)$, $\gamma_2 \sim [0, 1]$. We assume that the marked point $z = 0$ lies inside the closed contour $\gamma_1(z) \gamma_2(z + \tau) \gamma_1^{-1}(z + 1) \gamma_2^{-1}(z)$. The space \mathcal{R} (A.9) is called the Higgs bundle with the *quasi-parabolic structure* at the marked point $z = 0$.

The canonical transformations of (A.10) are (A.2) along with

$$\eta_\alpha \rightarrow f(z) \eta_\alpha(z, \bar{z}) f^{-1}(z), \quad h \rightarrow h f(0). \quad (\text{A.11})$$

The transformations are generated by the following first class constraints. Let $\Phi(z)$ be a meromorphic one-form on E_τ . Then

$$\eta_\alpha = \Phi(z), \quad \text{Res}(\Phi(z))|_{z=0} = p, \quad p \in \mathcal{O}. \quad (\text{A.12})$$

The form Φ is the so-called *the Higgs field*. Moreover, the constraints imply the quasi-periodicity of η_α

$$\eta_1(z, \bar{z}) = g_1(z) \eta_1(z + 1, \bar{z} + 1) g_1^{-1}(z), \quad \eta_2(z, \bar{z}) = g_2(z) \eta_2(z + \tau, \bar{z} + \bar{\tau}) g_2^{-1}(z). \quad (\text{A.13})$$

Let $\mu_j d\bar{z} \in \Omega^{(-j,1)}(E_\tau)$ be $(-j, 1)$ -differentials on E_τ . We choose the representatives from $\Omega^{(-j,1)}(E_\tau)$ that form a basis in the cohomology space $h^1(E_\tau, \Gamma^j)$, ($\dim h^1 = j$)

$$\mu_j = (\mu_{0,j} \partial_z^{j-1}, \mu_{2,j} \partial_z^{j-1}, \dots, \mu_{j,j} \partial_z^{j-1}). \quad (\text{A.14})$$

The coefficients $\mu_{s,j}$ coincide with the basis f_s (B.22) ($\mu_{s,j} = f_s$). The integrals

$$I_{s,j} = \int_{E_\tau} \text{tr}(\Phi^j) \mu_{s,j} d\bar{z}. \quad (\text{A.15})$$

are gauge invariant. They play the role of the Hamiltonians in the integrable hierarchy. The equations of motion on the phase space \mathcal{R} with respect to the Hamiltonian $I_{s,j}$ take the form

$$\partial_{s,j} \Phi = 0, \quad (\text{A.16})$$

$$(\partial_{s,j} g_\alpha) g_\alpha^{-1} = \Phi^{j-1} \mu_{s,j}, \quad (\text{A.17})$$

$$\partial_{s,j} p = 0.$$

Consider the symplectic quotient $\mathcal{R}^{red} = \mathcal{R}/\mathcal{G}_N$. Let $f(z)$ be the gauge transform that brings the transition function in the standard form ((A.5) for $d = 0$ and (A.6) for $d = 1$). The Lax operator is the corresponding gauge transform of the Higgs field Φ

$$L_N(z) = f(z) \Phi(z) f^{-1}(z). \quad (\text{A.18})$$

Then the first equation (A.16) is equivalent to the Lax equation

$$\partial_{s,j} L_N = [L_N, M_{N;s,j}], \quad (\text{A.19})$$

where $M_{N;s,j} = f^{-1} \partial_{s,j} f$.

In terms of the Lax matrix the integrals (A.15) have the form

$$I_{s,j} = \oint \text{tr}(L_N^{(0)})^j \mu_{s,j}. \quad (\text{A.20})$$

They can be found from the expansion on the basis of the elliptic functions

$$\text{tr}(L_N)^j(z) = I_{0,j} + \sum_{s=2}^N I_{s,j} \wp^{(s-2)}(z) \quad (\wp^{(k)}(z) = \partial_z^k \wp(z)). \quad (\text{A.21})$$

9.2 Appendix B. Elliptic functions.

We summarize the main formulae for elliptic functions, borrowed mainly from [30] and [31]. We consider the elliptic curve

$$E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, \quad q = \mathbf{e}(\tau) = \exp 2\pi i \tau. \quad (\text{B.1})$$

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n(n+1)\tau + 2nz)} = \quad (\text{B.2})$$

$$q^{\frac{1}{8}} e^{-\frac{i\pi}{4}} (e^{i\pi z} - e^{-i\pi z}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2i\pi z})(1 - q^n e^{-2i\pi z}).$$

The Weierstrass functions

$$\sigma(z|\tau) = \exp(\eta z^2) \frac{\vartheta(z|\tau)}{\vartheta'(0|\tau)}, \quad (\text{B.3})$$

where

$$\eta(\tau) = -\frac{1}{6} \frac{\vartheta'''(0|\tau)}{\vartheta'(0|\tau)}. \quad (\text{B.4})$$

$$\zeta(z|\tau) = \partial_z \log \vartheta(z|\tau) + 2\eta(\tau)z, \quad \zeta(z|\tau) \sim \frac{1}{z} + O(z^3). \quad (\text{B.5})$$

$$\wp(z|\tau) = -\partial_z \zeta(z|\tau). \quad (\text{B.6})$$

$$\wp(u; \tau) = \frac{1}{u^2} + \sum'_{j,k} \left(\frac{1}{(j+k\tau+u)^2} - \frac{1}{(j+k\tau)^2} \right). \quad (\text{B.7})$$

The next important function is

$$\phi(u, z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \quad (\text{B.8})$$

It has a pole at $z = 0$ and

$$\phi(u, z) = \frac{1}{z} + \zeta(u|\tau) + 2\eta(\tau)u + \frac{z}{2}((\zeta(u|\tau) + 2\eta(\tau)u)^2 - \wp(u)) + \dots, \quad (\text{B.9})$$

and

Relation to the Weierstrass functions

$$\phi(u, z)^{-1} \partial_u \phi(u, z) = \zeta(u+z) - \zeta(u) + 2\eta(\tau)z. \quad (\text{B.10})$$

$$\phi(u, z) = \exp(-2\eta_1 uz) \frac{\sigma(u+z)}{\sigma(u)\sigma(z)}, \quad (\text{B.11})$$

$$\phi(u, z)\phi(-u, z) = \wp(z) - \wp(u). \quad (\text{B.12})$$

Particular values

$$\wp'(z) = 0 \quad \text{for } z = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}. \quad (\text{B.13})$$

Addition formula. (Calogero functional equation.)

$$\phi(u, z)\partial_v \phi(v, z) - \phi(v, z)\partial_u \phi(u, z) = (\wp(v) - \wp(u))\phi(u+v, z). \quad (\text{B.14})$$

In fact, $\phi(u, z)$ satisfies more general relation which follows from the Fay three-section formula

$$\phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0. \quad (\text{B.15})$$

A particular case of this formula is (B.12) and

$$\phi(u_1, z)\phi(u_2, z) - \phi(u_1 + u_2, z)(\zeta(u_1) + \zeta(u_2) - 2\eta(\tau)(u_1 + u_2)) + \partial_z \phi(u_1 + u_2, z) = 0. \quad (\text{B.16})$$

It follows from (B.10), (B.12), (B.16) that for $u_1 + u_2 + u_3 = 0$

$$\phi(u_1, z)\phi(u_2, z)\phi(u_3, z) = [\wp(z) - \wp(u_3)] [\zeta(u_1) + \zeta(u_2) + \zeta(u_3 - z) + \zeta(z)]. \quad (\text{B.17})$$

Then

$$\phi(u_1, z)\phi(u_2, z)\phi(u_3, z)|_{z \rightarrow 0} = \frac{1}{z^3} + \frac{1}{z^2} [\zeta(u_1) + \zeta(u_2) + \zeta(u_3)] \quad (\text{B.18})$$

$$-\frac{1}{2}\wp'(u_3) - \wp(u_3) [\zeta(u_1) + \zeta(u_2) + \zeta(u_3)] + O(z)$$

Basis of elliptic functions on E_τ .

We consider elliptic functions on E_τ with poles at $z = 0$. Any elliptic meromorphic function $F(z)$ is represented in the form

$$F(z) = \sum_{j=0,2,3,\dots} c_j e^j, \quad (\text{B.19})$$

where

$$e^0 = 1, \quad e^j = \partial_z^{(j-2)} E_2(z). \quad (\text{B.20})$$

The dual basis f_k with respect to the pairing

$$\langle * | * \rangle = \int_{E_\tau}, \quad \langle f_k | e^j \rangle = \delta_k^j \quad (\text{B.21})$$

has the form

$$\begin{aligned} f_0 &= (\bar{z} - z)(1 - \chi(z, \bar{z})), \\ f_k &= z^{k-1} \chi(z, \bar{z}), \quad k > 1, \end{aligned} \quad (\text{B.22})$$

where $\chi(z, \bar{z})$ is a characteristic function of a small neighborhood \mathcal{U}_0 of $z = 0$

$$\chi(z, \bar{z}) = \begin{cases} 1, & z \in \mathcal{U}_0, \mathcal{U}'_0 \supset \mathcal{U}_0 \\ 0, & z \in E_\tau \setminus \mathcal{U}'_0. \end{cases} \quad (\text{B.23})$$

Theta functions with characteristics.

For $a, b \in \mathbb{Q}$ put :

$$\theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, \tau) = \sum_{j \in \mathbb{Z}} \mathbf{e} \left((j+a)^2 \frac{\tau}{2} + (j+a)(z+b) \right). \quad (\text{B.24})$$

In particular, the function ϑ (B.2) is the theta function with a characteristic

$$\vartheta(x, \tau) = \theta \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (x, \tau). \quad (\text{B.25})$$

For the simplicity we denote $\theta \left[\begin{smallmatrix} a/2 \\ b/2 \end{smallmatrix} \right] = \theta_{ab}$, ($a, b = 0, 1$).

References

- [1] A.G. Reyman, M.A. Semenov-Tyan-Shanskii, *Lie algebras and Lax equations with spectral parameter on elliptic curve*, Notes from Sci. Seminar LOMI Vol. **150**, (1986), 104–118 (in Russian).
- [2] A. Levin, M. Olshanetsky and A. Zotov, *Hitchin Systems - symplectic maps and two-dimensional version*, **nlin.SI/0110045**.
- [3] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. Jour. **54** (1987), 91–114.

- [4] N. Nekrasov, *Holomorphic Bundles and Many-Body Systems*, Commun.Math.Phys. 180 (1996), 587-604.
- [5] E.Markman, *Spectral curves and integrable systems*, Comp. Math. **93**, (1994) 255-290.
- [6] K. Ueno, K. Takasaki, *Toda lattice hierarchy*, in "Group Representations and Systems of Differential Equations", K. Okamoto ed., Adv. Studies in Pure Math. **4** (North-Holland/Kinokunia 1984), 1-95.
- [7] R. Kashaev, M. Saveliev, S. Savelieva and A. Vershik, *On nonlinear equations associated with Lie algebras of diffeomorphism groups of two-dimensional manifolds.*, Ideas and methods in mathematical analysis, stochastics, and applications Oslo, 1988), 295–307, Cambridge Univ. Press, Cambridge, (1992).
- [8] K. Takasaki and T. Takebe, *SDiff(2) Toda equation - hierarchy, tau function and symmetries*, Lett. Math. Phys. **23**, (1991) 205-214.
- [9] M. Olshanetsky, *Integrable tops and non-commutative torus*, talk at the XVI Max Born symposium "Supersymmetries and Quantum Symmetries", ed. E.Ivanov et al, Dubna 2002.
- [10] V.I. Arnold and B.A. Khesin, *Topological methods in Hydrodynamics*, Applied Mathematical Science, 125. Springer-Verlag, New York, (1998).
- [11] L.A. Dikii, *Hamiltonian systems connected with the rotation group*, Funct. Anal. Appl. **6** (1972), 326-327
- [12] S.V. Manakov, *Note on integration of Euler's equation of the dynamics of n-dimensional rigid body*, Funct. Anal. Appl., **10** (1976) 328–329.
- [13] B.A. Dubrovin, I.M. Krichever and S.P. Novikov, *Integrable systems. I*, Current problems in mathematics. Fundamental directions, Vol. 4, 179–284, 291, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, (1985) (in Russian).
- [14] S.L. Ziglin, *Non-integrability of the problem of the motion of four point vortices*, Dokl. Akad. Nauk SSSR **250**, no. 6, (1980) 1296–1300.
- [15] M.I. Golenishcheva-Kutuzova and A.G. Reyman, *Integrable equations that are connected with a Poisson algebra*, Zap. Nauchn. Sem. LOMI **169** (1988) 44 (in Russian).
- [16] K.Takasaki, *Area-Preserving Diffeomorphisms and Nonlinear Integrable Systems*, hep-th/9112041.
- [17] D.R. Lebedev and Yu.I. Manin, *Conservation laws and Lax representation of Benney's long wave equations*, Phys. Rev, A **74** (1979) 154–156;
Benney's long wave equations. II, Zap. Nauchn. Sem. LOMI **96** (1980) 169–178 (in Russian).
- [18] V.E. Zakharov, *On the Benny's equation*, Physica **3D** (1981), 193-202.
- [19] K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. **7** (1995), 743–808.
- [20] R.S. Ward, *Infinite-dimensional gauge groups and special nonlinear gravitons*, J. Geometry and Physics, **8**, (1992) 317–325.

- [21] D.B Fairlie, P. Fletcher, C.K. Zachos, *Infinite Dimensional Algebras and a trigonometrical basis for the classical Lie algebras*, J. Math. Phys.**31**, (1990) 1088.
- [22] V. Zeitlin *Algebraization of 2-D ideal fluid hydrodynamical systems and their finite-mode approximations*, Advances in turbulence, 3 (Stockholm, 1990), 257–260, Springer, Berlin, 1991.
- [23] J. Hoppe, M. Olshanetsky and S. Theisen, *Dynamical systems on quantum tori Lie algebras*, Commun.Math.Phys. **155**, (1993) 429–448.
- [24] F. Calogero, J.Math.Phys. **12** (1971), 419.
- [25] J. Moser, Adv.Math. **16** (1975) 197–220.
- [26] A. Gorsky, N. Nekrasov, *Elliptic Calogero-Moser system from two dimensional current algebra*, **hep-th/9401021**.
- [27] I. Krichever, *Vector bundles and Lax equations on algebraic curves*, Commun. Math. Phys. **229**, No.2, (2002) 229-269.
- [28] A. Polishchuk and A. Schwarz, *Categories of holomorphic vector bundles on noncommutative two-tori*, **math.QA/0211262**.
- [29] K. Takasaki, *Anti-self-dual Yang-Mills equations on noncommutative spacetime*, J. Geom. Phys. **37**, No.4 (2001) 291-306.
- [30] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer-Verlag, Berlin-Heidelberg, 1976.
- [31] D. Mumford, *Tata Lectures on Theta I, II*, Birkhäuser Boston, 1984.